**Problem set 8**

April 2, 2010

Due date: *Need not submit (in view of the class text on 10th April)*

**Exercise 65.** Let $X_n$ be independent and $\mathbf{P}(X_n = n^a) = \frac{1}{2} = \mathbf{P}(X_n = -n^a)$ where $a > 0$ is fixed. For what values of $a$ does the series $\sum X_n$ converge a.s.? For which values of $a$ does the series converge absolutely, a.s.?

**Exercise 66.** (**Random series**) Let $X_n$ be i.i.d $N(0,1)$ for $n \geq 1$.

(1) Show that the random series $\sum X_n \frac{\sin(n\pi t)}{n}$ converges a.s., for any $t \in \mathbb{R}$.
(2) Show that the random series $\sum X_n \frac{t^n}{\sqrt{n!}}$ converges for all $t \in \mathbb{R}$, a.s.

[**Note:** The location of the phrase "a.s" is all important here. Let $A_t$ and $B_t$ denote the event that the series converges for the fixed $t$ in the first or second parts of the question, respectively. Then, the first part is asking you to show that $\mathbf{P}(A_t) = 1$ for each $t \in \mathbb{R}$, while the second part is asking you to show that $\mathbf{P}(\cap_{t\in\mathbb{R}} B_t) = 1$. It is also true (and very important!) that $\mathbf{P}(\cap_{t\in\mathbb{R}} A_t) = 1$ but showing that is not easy.]

**Exercise 67.** Suppose $X_n$ are i.i.d random variables with finite mean. Which of the following assumptions guarantee that $\sum X_n$ converges a.s.?

(1) (i) $\mathbf{E}[X_n] = 0$ for all $n$ and (ii) $\sum \mathbf{E}[X_n^2 \wedge 1] < \infty$.
(2) (i) $\mathbf{E}[X_n] = 0$ for all $n$ and (ii) $\sum \mathbf{E}[X_n^2 \wedge |X_n|] < \infty$.

**Exercise 68.** (**Large deviation for Bernoullis**). Let $X_n$ be i.i.d Ber$(1/2)$. Fix $p > \frac{1}{2}$.

(1) Show that $\mathbf{P}(S_n > np) \leq e^{-np\lambda} \left(\frac{e^\lambda+1}{2}\right)^n$ for any $\lambda > 0$.
(2) Optimize over $\lambda$ to get $\mathbf{P}(S_n > np) \leq e^{-nI(p)}$ where $I(p) = -p\log p - (1-p)\log(1-p)$. (Observe that this is the *entropy* of the Ber$(p)$ measure introduced in the first class test).
(3) Recall that $S_n \sim \text{Binom}(n, 1/2)$, to write $\mathbf{P}(S_n = \lceil np \rceil)$ and use Stirling's approximation to show that
$$\mathbf{P}(S_n \geq np) \geq \frac{1}{\sqrt{2\pi np(1-p)}} e^{-nI(p)}.$$
(4) Deduce that $\mathbf{P}(S_n \geq np) \approx e^{-nI(p)}$ for $p > \frac{1}{2}$ and $\mathbf{P}(S_n < np) \approx e^{-nI(p)}$ for $p < \frac{1}{2}$ where the notation $a_n \approx b_n$ means $\frac{\log a_n}{\log b_n} \to 1$ as $n \to \infty$ (i.e., asymptotic equality on the logarithmic scale).

**Exercise 69.** Carry out the same program for i.i.d exponential(1) random variables and deduce that $\mathbf{P}(S_n > nt) \approx e^{-nI(t)}$ for $t > 1$ and $\mathbf{P}(S_n < nt) \approx e^{-nI(t)}$ for $t < 1$ where $I(t) := t - 1 - \log t$.

**Exercise 70.** Let $Y_1, \ldots, Y_n$ be independent random variables. A random variable $\tau$ taking values in $\{1, 2, \ldots, n\}$ is called a *stopping time* if the event $\{\tau \leq k\} \in \sigma(Y_1, \ldots, Y_k)$ for all $k$ (equivalently $\{\tau = k\} \in \sigma(Y_1, \ldots, Y_k)$ for all $k$).

(1) Which of the following are stopping times? $\tau_1 := \min\{k \leq n : S_k \in A\}$ (for some fixed $A \subset \mathbb{R}$). $\tau_2 := \max\{k \leq n : S_k \in A\}$. $\tau_3 := \min\{k \leq n : S_k = \max_{j \leq n} S_j\}$. In the first two cases set $\tau = n$ if the desired event does not occur.
(2) Assuming each $X_k$ has zero mean, show that $\mathbf{E}[S_\tau] = 0$ for any stopping time $\tau$. Assuming that each $X_k$ has zero mean and finite variance, show that $\mathbf{E}[S_1^2] \leq \mathbf{E}[S_\tau^2] \leq \mathbf{E}[S_n^2]$ for any stopping time $\tau$.
(3) Give examples of random $\tau$ that are not stopping times and for which the results in the second part of the question fail.